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SETS WITH A MODE

BY

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ABSTRACT

Let M be a point and S be a compact set in R^2 such that S is the closure of its interior. The theorem desired says that if M is a mode of S then S is convex and centrally symmetric with respect to M. Some conditions on the boundary of S are needed for the proof given.

Thoughout this paper S will be a nonempty compact subset of R^2 which is the closure of its interior and M will be a mode of S (defined below). In their paper Dharmadhikari and Jogdeo [1] prove that S is convex and hence centrally symmetric with respect to M provided S has Jordan polygonal boundary. The aim of this paper is to replace the condition of a Jordan polygonal boundary with a condition satisfied by all compact convex sets. The condition for this paper is that the boundary of S will consist of a finite number of acceptable closed curves (defined below) which meet in at most a finite number of points.

DEFINITIONS. For any real number t and any unit vector u in R^2, let $L(u,t)$ be the line

$$\{z \in R^2 : \langle u, z - M \rangle = t\}$$

and m be Lebesgue measure on the line. M is a *mode* of S if, for each u, $m(L(u,t) \cap S)$ is a nonincreasing function of t for $t \geqq 0$ and a nondecreasing function of t for $t \leqq 0$.

A curve is an *acceptable closed curve* if there is a homeomorphism f of the unit circle ($[0,2\pi]$ with 0 and 2π identified) onto the curve such that f has a nonzero left derivative f'_L everywhere on $(0,2\pi]$, f has a nonzero right derivative f'_R everywhere on $[0,2\pi)$, f'_L is continuous from the left, f'_R is continuous from the right and $f'_L = f'_R = f'$ except for at most a countable number of points.

The purpose of this paper is to prove the following

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THEOREM. *Let S be a nonempty compact subset of R^2 which is the closure of its interior. Suppose S has a mode M and the boundary of S consists of a finite number of acceptable closed curves which meet in at most a finite number of points. Then S is convex and centrally symmetric with respect to M.*

Before proceeding with the proof, there are a few consequences of the condition on the boundary of S that should be noted to give a more geometric idea of what an acceptable closed curve is and how the condition will be used. First the existence of nonzero left and right derivatives at a point imply the existence of tangent rays at that point. If $f'_l = f'_R$ the tangent rays are the two opposite rays of the tangent line.

Second, if f is the homeomorphism guaranteed by the definition, then

$$f(b) - f(a) = \int_a^b f'(x)dx, \qquad 0 \leqq a < b \leqq 2\pi.$$

This may be concluded from exercise 18:41d in Hewitt and Stromberg [2] or from 8:11 (or the proof of 8:21) in Rudin [6].

Third, the image of f has finite length given by, for example, $\int_0^{2\pi} |f'(x)| dx$ [3, p. 36]. Thus the boundary of S has finite length. No precise definition of length will be needed but elementary calculus such as Purcell 16:4 [4] will be used as will the fact that if one side of a rectangle intersects S in a length l greater than the opposite side then the length of the boundary of S inside the rectangle is at least l.

Finally, it follows from general knowledge (or [5, 24.1]) that the boundary of a compact convex set with nonempty interior is an acceptable closed curve. The proof of the Theorem now follows with some notation and a sequence of lemmas.

NOTATION. For any angle t, let

$$R(t) = \{x \in R^2 : x = M + a(\cos t, \sin t),\ a > 0\}.$$

If x in the boundary of S has a tangent line denote it by $T(x)$. If $x \neq y$, let (x, y) be the open line segment between x and y. If an endpoint is to be included, a square bracket will replace the appropriate parenthesis. Let ∂S denote the boundary of S and d be Euclidean distance in R^2.

LEMMA 1. $m\{t \in [0, 2\pi] : R(t) \subset T(x)$ *for some* $x \in \partial S\} = 0$.

PROOF. For any t in $(0, 2\pi)$, let $s(t)$ be the length of the boundary of S within the angle from $R(0)$ to $R(t)$. Then s is a monotone increasing function so

$s(2\pi) \geqq \int_0^{2\pi} s'(t)dt$. This implies that $m\{t \in (0, 2\pi): s'(t) = \infty\} = 0$. But if $R(t) \subset T(x)$ for some $x \neq M$ in the boundary of S, then $s'(t) = \infty$. Since $T(M)$ includes $R(t)$ for at most a finite number of angles, the conclusion follows.

LEMMA 2. *S is star-shaped with respect to M.*

PROOF. Since S is the closure of its interior, $\{x \in \partial S : (x, M)$ is not contained in $S\}$ is a subset of the closure of its interior. Therefore, if S is not star-shaped with respect to M, it contains a nonempty open interval (a, b). For $a < t < b$, there is a $z(t)$ in $R(t) \cap \partial S$, $z(t) \neq M$, with $z(t)$ not in the closure of $((z(t), M) \cap S)$. For all but a countable number of such t's, $T(z(t))$ exists. For all but a set of measure 0, $M \not\in T(z(t))$. Therefore there is a point z in ∂S such that z is on only one acceptable closed curve, $T(z)$ exists, $M \not\in T(z)$ and z is not in the closure of $((z, M) \cap S)$. By the continuity assumptions of f' and thus on T there is a neighborhood V of z such that if $y \in V \cap \partial S$ and $T(y)$ exists, then $M \not\in T(y)$ and y is not in the closure of $((y, M) \cap S)$. V contains a ball B centered at z of radius $r > 0$. Measuring angles counterclockwise, let $L(a)$ be the line through z making the angle a with $T(z)$. There is a $b > 0$ such that $M \not\in L(a)$ for $|a| \leqq b$. Define $s(a)$ to be the length of ∂S strictly in the small angles between $L(a)$ and $T(z)$ but outside B.

For $a > 0$, $s(a)$ is a monotone increasing function and therefore s' exists almost everywhere and

$$s(b) \geqq \int_0^b s'(a)\,da.$$

But, if it exists, $s'(a) \geqq r \cot a$. To see this, let $0 < a < b$ be given such that $s'(a)$ exists. Fix c with $0 < c < a$. For any $h > 0$, let $L_1(h)$ be the line parallel to $L(a)$ on the side of $L(a)$ not containing M through a point of intersection of $L(a+h)$ with the boundary of B. Let $L_2(h)$ be the parallel through the other point of intersection so that $L_2(h)$ is on the same side of $L(a)$ as M. For any point x on L_2 the corresponding point on L_1 will be the intersection of the perpendicular to L_2 through x with L_1. For h small enough, $L_1(h)$ and $L_2(h)$ intersect $L(c)$ and $L(-c)$ closer to z than does ∂S and therefore the portion of L_1 between $L_1 \cap L(-c)$ and the point on L_1 corresponding to $L(-c) \cap L_2$ is in S while the corresponding line segment on L_2 is outside S. (Fig. 1).

Beeause

$$s(a + h/2) - s(a - h/2) \geqq m(L_1 \cap B \cap S) - m(L_2 \cap B \cap S),$$

$$s(a + h/2) - s(a - h/2) \geqq 2r \sin(h/2) \cot(a + c)$$

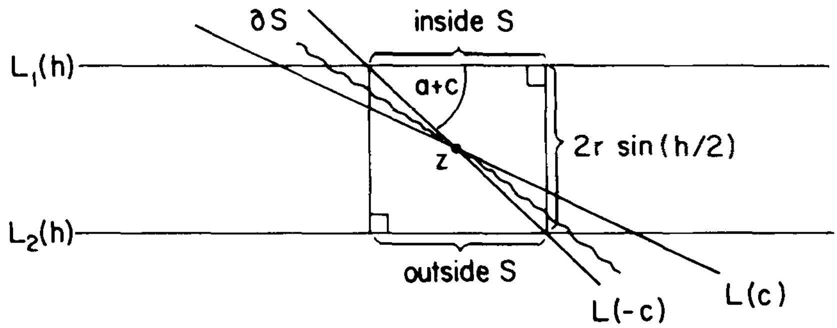

Fig. 1

and therefore $s'(a) \geqq r\cot(a+c)$. As c was an arbitrary number in $(0, a)$,

$$s'(a) \geqq r\cot a \quad \text{and}$$

$$s(b)\Big| \geqq \int_0^b r\cot a\, da = \infty.$$

This contradicts the finite length of ∂S and the proof is complete.

It follows from Lemma 2 that the finite number of acceptable closed curves intersect only at M if anywhere.

Lemma 3. *If $R(t) \cap S$ is nonempty, it contains a unique point, $r(t)$, which is also in the boundary of S. Furthermore, if $r(t)$ is defined to be M when $R(t) \cap S$ is empty then r is continuous.*

Proof. Lemma 2 implies M is in S so if $R(t) \cap S$ is empty then M is in the boundary of S. Now, for each t such that $R(t) \cap S$ is not empty choose $x(t)$ in $R(t) \cap dS$. Otherwise, let $x(t) = M$. For fixed t_1, $\lim_{t<t_1} x(t)$ exists. This can be seen as follows. If not, let

$$t_2 < t_3 < \cdots < t_{2n} < t_{2n+1} < \cdots$$

be such that $t_n \to t_1$ and $\lim x(t_{2n}) \neq \lim x(t_{2n+1})$. Because S is star-shaped with respect to M, the length of the boundary of S is greater than

$$\sum_{n=1}^{\infty} d(x(t_{2n}), x(t_{2n+1}))$$

which is infinite. Similarly,

$$\lim_{t>t_1} x(t) \text{ exists.}$$

Let $x = \lim_{t<t_1} x(t)$ and $y = \lim_{t>t_1} x(t)$ and suppose $d(x, M) < d(y, M)$. Let $z = \lim_{t<t_1+\pi} x(t)$.

For each t, let $A(t)$ be the line through $x(t)$ and M and $B(t)$ be the parallel line through y. Then $m(B(t)\cap S)\leqq m(A(t)\cap S)$ since M is a mode. But

$$\lim_{t<t_1} m(B(t)\cap S) = d(y,z) > d(x,z) = \lim_{t<t_1} m(A(t)\cap S)$$

since S is the closure of its interior, star-shaped with respect to M and at most a finite number of acceptable closed curves in ∂S meet at M. A similar contradiction follows from $d(x,M) > d(y,M)$ and the proof is complete.

NOTATION. Let $\rho(t) = d(r(t), M)$. Since the interior of S is nonempty, there is an $r_1 > 0$ such that $\rho(t) + \rho(t+\pi) > r_1$.

LEMMA 4. *For any $t_1 < t_2$, there is a t in (t_1, t_2) with $r(t) \neq M$.*

PROOF. If not we may choose t_0, t_1 and t_2 such that $r(t) = M$ for $t_1 \leqq t \leqq t_2$ but $r(t) \neq M$ for $t_0 < t < t_1$. Now $\rho(t_1+\pi) > r_1$. Choose η, $0<\eta<t_2 - t_1$ so that $|t-t_1|<\eta$ implies $\rho(t+\pi)>r_1$. For $0<h$, let $L_1(h)$ be the line through $r(t+h+\pi)$ parallel to the line through $r(t)$ and M. With $t_1 < t < t_2$, by perhaps increasing h slightly, we may assume $r(t+h+\pi)$ is the closest point to M on $L_1 \cap \partial S$ between $R(t_1+\pi)$ and $R(t_2+\pi)$. Let $t(h)$ be the minimum angle t between t_0 and t_1 such that $r(t)$ is in L_1. As h goes to 0, $t(h)$ will tend to t_1. For h small, $m(L_1(h)\cap S) \geqq d_1 + d_2$ (Fig. 2), where

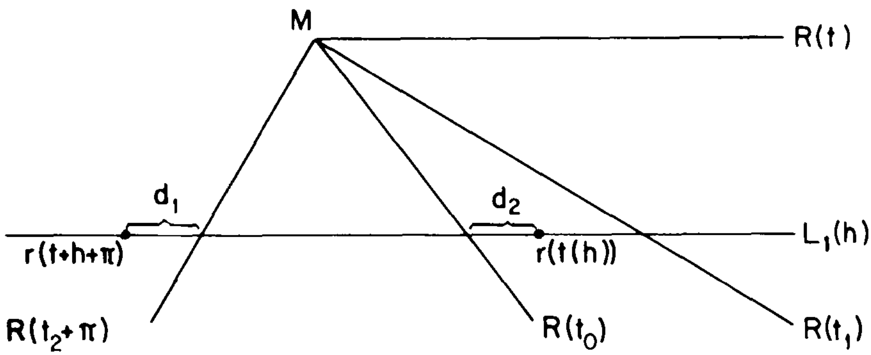

Fig. 2

$$d_1 = \rho(t+h+\pi)\cos(h) - \rho(t+h+\pi)\sin(h)\cot(t_2 - t)$$

and

$$d_2 = \rho(t(h))\cos(t - t(h)) - \rho(t+h+\pi)\sin(h)\cot(t - t_0).$$

Now, $\rho(t(h))\cos(t-t(h)) = \rho(t+h+\pi)\sin(h)\cot(t-t(h))$ and $\rho(t+\pi) \geqq m(L_1(h)\cap S)$. Thus

$$\rho(t+\pi)-\rho(t+h+\pi)\cos(h)$$
$$\geqq \rho(t+h+\pi)\sin(h)[\cot(t-t(h))-\cot(t_2-t)-\cot(t-t_0)].$$

For t sufficiently close to t_1 and h sufficiently small, $\rho(t+\pi)-\rho(t+h+\pi)\geqq r_1 h\cot(t-t_1)$ and so $|\rho'(t+\pi)|\geqq r_1\cot(t-t_1)$. Hence, the length of the boundary of S is at least

$$\int_{t_1}^{t_1+\varepsilon} r_1\cot(t-t_1)\,dt$$

(see, for example, Purcell [4], p. 528) which is infinite.

Lemma 5. *Suppose the boundary of S has tangent lines at $r(t)$ and $r(t+\pi)$. If M is not in $T(r(t))\cup T(r(t+\pi))$, then $T(r(t))$ is parallel to $T(r(t+\pi))$.*

Proof. Suppose not. Let x be the intersection of $T(r(t))$ with $T(r(t+\pi))$ and a be the angle between them. Let $L(0)$ be the line through $r(t)$ and M. For $h>0$ let $L(h)$ be the line parallel to $L(0)$ at distance h from M such that M and x are on the same side of $L(h)$. It follows from Lemma 4 that

$$m(L(h)\cap S)-m(L(0)\cap S)\geqq h\tan a+\sigma(h)$$

where $\sigma(h)/h$ tends to 0 with h. For h small enough this implies that $m(L(h)\cap S)>m(L(0)\cap S)$ which contradicts the fact that M is a mode of S.

Corollary 1. *If $\rho(t)$ is decreasing (increasing) for $a<t<b$, then so is $\rho(t+\pi)$.*

Proof. Lemma 5 guarantees that if $\rho'(t)$ is negative then so is $\rho'(t+\pi)$. Thus the conclusion is valid if $\rho'(t)$ exists. The continuity conditions in the definition of an acceptable closed curve are then sufficient to guarantee the conclusion when $\rho'(t)$ does not exist. The example after Corollary 2 illustrates why the continuity assumptions are needed for this proof.

Corollary 2. *M is not in the boundary of S.*

Proof. If $M=r(t_1)$, the continuity conditions in the definition of an acceptable closed curve and Lemma 4 imply that there is a $t_0<t_1$ and a $t_2>t_1$ such that $\rho(t)$ is decreasing in (t_0,t_1) and increasing in (t_1,t_2). Using Corollary 1, it is clear that lines close to M parallel to $R(t_1+\pi)$ will intersect S in a segment longer than $\rho(t_1+\pi)$. This contradiction establishes the result.

Example. Let f be the usual Cantor ternary function. Define

$$\rho(t)=f(t/\pi)\quad\text{for}\quad 0\leqq t\leqq\pi\quad\text{and}$$

$$\rho(t)=f((2\pi-t)/\pi)\quad\text{for}\quad \pi\leqq t\leqq 2\pi.$$

Then $(t, \rho(t))$ give polar coordinates for the boundary of a set S. Whenever ∂S has a tangent line at $(t, \rho(t))$, it also has the same or a parallel tangent line at $(t+\pi, \rho(t+\pi))$. However ρ is increasing for $0 \leqq t \leqq \pi$ and decreasing for $\pi \leqq t \leqq 2\pi$. Thus, in some sense, it is possible to have a set satisfying Lemma 5 but not Corollary 1 or 2. The origin is not a mode of S so this example only shows that the continuity assumptions are needed for the proof of Corollary 1 and not that they are needed for the final theorem. What is needed for the final theorem is a question I haven't been able to resolve.

LEMMA 6. *S is convex.*

PROOF. Suppose not. Choose $a < b < a + \pi$ such that $(r(a), r(b)) \not\subset S$. Now let L be the line closest to M which is parallel to the line through $r(a)$ and $r(b)$ but still contains some point $r(t)$ for $a < t < b$. Let c be the infimum of $\{t \in (a,b) : r(t) \in L\}$ so that $(r(t), M) \cap L$ is not empty for t in (a, c) but $r(c) \in L$. Let L_1 be the parallel to L through $r(c+\pi)$. It follows from Lemma 5 and the continuity conditions on an acceptable closed curve that $[r(t), M) \cap L_1$ is not empty for $a + \pi \leqq t \leqq b + \pi$. Hence, for t in (a, c) and close to c, the line through $r(t)$ parallel to $R(c)$ intersects S in a segment longer than $[r(c), r(c + \pi)]$. This contradicts the fact that M is a mode of S.

PROOF OF THE THEOREM. Lemma 6 says S is convex. If we choose a so that $\rho(a) = \rho(a+\pi)$ central symmetry follows since

$$\rho(b) - \rho(a) = \int_a^b \rho'(t)dt = \int_{a+\pi}^{b+\pi} \rho'(t)dt = \rho(b+\pi) - \rho(a+\pi).$$

Alternatively, Theorem 2 in Dharmadhikari and Jogdeo [1] says that if a compact convex body has a mode then it is centrally symmetric about the mode.

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